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Technical Note

NOTE ON THE TIME-DEPENDENT RELATIONSHIP BETWEEN EMISSIONS OF GREENHOUSE GASES AND CLIMATE CHANGE

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1. Introduction

The relationship between the net anthropogenic emissions of greenhouse gases (emissions[1]) and the resulting change in climate is relevant for several reasons.

The international treaties dealing with the mitigation of climate change are such that countries will be able to achieve their quantitative emission limitation and reduction objectives through measures limiting the emission of different greenhouse gases. It is necessary then to have a metric that allows the addition of emissions of different greenhouse gases.

The evaluation of the relative responsibility of different countries requires the estimation of the change in climate resulting from emissions from different sources over different time periods.

Government and private sector policy makers are faced with the choice among alternative strategies which result in a change in the mix of greenhouse gas emissions over time. This choice requires a tool to estimate the result of each alternative in terms of the future climate.

This note approaches the problem of establishing the time-dependent relationship between emissions and climate change by reducing the complex dependence of the increase in global mean surface temperature (temperature increase[2]) upon emissions to the simplest possible expression.

It is assumed that the temperature increase ΔT at time t, as a function of the past emissions $e(t')$ and of all other variables $\vec{x}$, is invariant with respect to the addition operation, that is:

$$\Delta T\,(e_1(t') + e_2(t'), \vec{x}, t) = \Delta T\,(e_1(t'), \vec{x}, t) + \Delta T(e_2(t'), \vec{x}, t) \qquad (1)$$

The acceptance of the concept of carbon dioxide equivalent emissions implies the acceptance of this assumption. It follows that, in particular, the emissions from different sources may be added for the same gas, since it is admitted for different gases. The important question is how to deal with the time dependence of the effect of emissions, since it is different for different greenhouse gases. The time dependence of the relationship between emissions and climate change is treated explicitly in this note.

The use of the temperature increase as a measure of climate change is not unique. The rise in mean sea level and the time rate of change of temperature are also global

[1] In this note, the word emissions is used, for the sake of brevity, to mean the net anthropogenic emissions of greenhouse gases, or the difference between anthropogenic emissions by sources and anthropogenic removals by sinks of greenhouse gases.

[2] In this note, the expression temperature increase is used, for the sake of brevity, to mean the increase in global mean surface temperature resulting from net anthropogenic emissions of greenhouse gases.

indicators of climate change. The rate of change of the temperature increase and the extension of the formulation to the mean sea-level rise are also considered in this note.

The Global Warming Potential (GWP) proposed by the Intergovernmental Panel on Climate Change (IPCC) is a weighting factor used for adding impulse emissions of different greenhouse gases so that they produce equivalent results in terms of temperature increase after a specified time lag. It is shown in this note that the IPCC GWP is a special case of a generalized global warming potential.

The proposal presented by the Government of Brazil for the Kyoto Protocol included, for illustration purposes, a “policy-maker” model relating emissions to the temperature increase. It is shown that the “policy-maker” model is also a special case of the general formulation.

2. Relationship between emissions, additional concentration, mean radiative forcing and temperature increase

The significant factors affecting the time dependence of the relationship between emissions and temperature increase are the decay of the additional atmospheric concentration of greenhouse gases (additional concentration[3]) and the transient adjustment of the temperature increase to a changed greenhouse gas concentration.

For the majority of the greenhouse gases, the time dependence of the additional concentration follows a simple exponential decay.

In the case of carbon dioxide, the complex decay of the additional concentration with time is approximated by a sum of exponentially decaying functions, one for each fraction of the additional concentrations.

For a constant additional concentration of a greenhouse gas, there is a linear relationship between this additional concentration and the long-term steady-state temperature increase. In order to consider the time dependence, however, it is necessary to consider the transient adjustment of the temperature increase to the additional concentration. Such adjustment is also approximated by a sum of exponential laws, with fractions corresponding to different time constants.

All other factors that determine the relationship between emissions and temperature increase are not ignored, but lumped into the constants.

Non-linearities, such as for instance the non-linear dependence of the infrared absorption cross-section of carbon dioxide upon the atmospheric concentration, are ignored and should not affect the relative conclusions obtained with the simplified formulation, regarding the relative importance of different gases or the relative contribution of different sources.

An impulse emission of a greenhouse gas does not result in an instantaneous increase of the same magnitude, due to the removal of a fraction of the emitted gas in a time scale shorter than the annual scale used. This fact is taken into account by stipulating a factor, which is applied to the emissions when computing the resulting additional concentration.

The time-dependent relationship between the emissions and the additional concentration of a greenhouse gas g is given, in its simplest form, by:

$$\Delta r_g(t) = b_g \int_{-\infty}^{t} e_g(t') \left[\sum_{r=1}^{R} f_{gr} \; e^{-(t-t')/t_{gr}} \right] dt' \qquad (2)$$

where:

[3] In this note, the expression additional concentration is used, for the sake of brevity, to mean the additional atmospheric concentration of greenhouse gases due to net anthropogenic emissions of such gases.

$\Delta r_g(t)$ is the additional concentration of greenhouse gas g resulting from emissions in previous times;

b_g is the increase in concentration of greenhouse gas g per unit of annual emission of that gas;

$e_g(t)$ is the annual emission of greenhouse gas g at time t;

R is the total number of fractions of the additional concentration;

t_{gr} is the exponential decay time constant of the rth fraction f_{gr} of the additional concentration of greenhouse gas g.

f_{gr} is the rth fraction of the additional concentration of greenhouse gas g, decaying exponentially with a time constant t_{gr}.

The constraint is imposed that:

$$\sum_{r=1}^{R} f_{gr} = 1 \qquad (3)$$

For carbon dioxide, the decay is approximated by 5 exponential functions (R=5); for all other greenhouse gases, a simple exponential decay is adopted (R = 1 and f_{g1} = 1).

An effective decay time constant $\bar{t}_g$ is defined as the weighted mean of the decay time constants:

$$\bar{t}_g = \sum_{r=1}^{R} f_{gr}\, t_{gr} \qquad (4)$$

The representation of the decay by a sum of exponential functions is only an empirical approximation to the observational data. There is thus no meaning to a single exponential decaying with the effective decay time constant $\bar{t}_g$. This definition is nevertheless useful as a constant in some of the expressions.

For greenhouse gases with exponential decay of the additional concentration, the fact that the emissions are specified as their annual values, implies a value of b different from one; indeed, if emissions are constant over a period of length ΔT, the additional concentration at the end of the period is:

$$\Delta r_g = e_g \int_0^{\Delta T} e^{-t/t_g}\, dt$$

$$= e_g\, t_g \left(1 - e^{-\Delta T / t_g}\right) \qquad (5)$$

or, for a period of one year and the time constant t_g expressed in years,

$$b_g = t_g \left(1 - e^{-1/t_g}\right) \tag{6}$$

The relationship between the additional concentration of greenhouse gas g and the resulting increase in mean radiative forcing is given by:

$$\Delta \overline{Q}_g(t) = \overline{s}_g \, \Delta r_g(t) \tag{7}$$

where:

$\Delta \overline{Q}_g(t)$ is the mean rate of deposition of energy on the earth's surface, or mean radiative forcing, per unit of additional concentration of greenhouse gas g;

$\overline{s}_g$ is the change in mean radiative forcing per unit of additional concentration of greenhouse gas g.

The time-dependent relationship between the mean radiative forcing and the resulting temperature increase can be approximated by considering the results of full climate models and fitting exponential functions to their results. Such results indicate that the temperature increase response to an instantaneous doubling of the carbon dioxide concentration and therefore of the mean radiative forcing can be approximated by a function of the type:

$$\Delta T_g(t) = \text{constant} \left[1 - \sum_{s=1}^{S} l_s \, e^{-t/t_{cs}}\right] \tag{8}$$

It follows that the response function to an impulse of additional concentration is its time derivative:

$$\Delta T_g(t) = \text{constant} \left[\sum_{s=1}^{S} \frac{l_s}{t_{cs}} \, e^{-t/t_{cs}}\right] \tag{9}$$

The time-dependent relationship between the mean radiative forcing and the resulting temperature increase is then given by:

$$\Delta T_g(t) = (1/C) \int_{-\infty}^{t} \Delta \overline{Q}_g(t') \left[\sum_{s=1}^{S} l_s \, (1/t_{cs}) \, e^{-(t-t')/t_{cs}}\right] dt' \tag{10}$$

where:

C is the heat capacity of the climate system;

S is the total number of fractions of the radiative forcing;

l_s is the sth fraction of the radiative forcing that reaches adjustment exponentially with a time constant t_{cs}.

The constraint is imposed that:

$$\sum_{s=1}^{S} l_s = 1 \tag{11}$$

t_{cs} is the exponential adjustment time constant of the sth fraction l_s of the temperature increase.

An effective temperature increase adjustment time constant $\bar{t}_c$ is defined as the inverse of the weighted mean of the inverse of the temperature increase adjustment time constants. Here, again, this concept is useful even though there is no meaning to an exponential function with this time constant.

$$\bar{t}_c = \frac{1}{\sum_{s=1}^{S} l_s \, (1/t_{cs})} \tag{12}$$

The combination of expressions (7) and (10) provide the relationship between the additional concentration of greenhouse gas g and the resulting temperature increase:

$$\Delta T_g(t) = (1/C)\, \bar{s}_g \int_{-\infty}^{t} \Delta r_g(t') \left[\sum_{s=1}^{S} l_s \, (1/t_{cs}) \, e^{-(t-t')/t_{cs}} \right] dt' \tag{13}$$

The combination of expressions (2) and (13) results in an expression relating the emissions of greenhouse gas g directly to the temperature increase:

$$\Delta T_g(t) = (1/C)\, \bar{s}_g \; b_g \int_{-\infty}^{t} \left\{ \int_{-\infty}^{t'} e_g(t'') \left[\sum_{r=1}^{R} f_{gr} \, e^{-(t'-t'')/t_{gr}} \right] dt'' \right\} \left[\sum_{s=1}^{S} l_s \, (1/t_{cs}) \, e^{-(t-t')/t_{cs}} \right] dt' \tag{14}$$

3. Normalized response functions

The relationships introduced in the preceding section can be expressed in terms of a constant, specific for each gas, multiplied by a normalized response function representing the time dependence. The normalization is different for each response function: the appropriate constant is chosen so that the normalized response functions for the different greenhouse gases are of similar magnitude.

The introduction of the normalized response functions allows the time-dependent portion of the relationship between two variables to be represented by the convolution of the independent variable with the normalized response function.

From emissions to additional concentration

The relationship between emissions and additional concentration in expression (2) can be written as:

$$\Delta r_g(t) = b_g \int_{-\infty}^{t} e_g(t')\, \Phi_g(t-t')\, dt' \tag{15}$$

where:

$$\Phi_g(t) = \sum_{r=1}^{R} f_{gr}\ \Phi_{gr}(t) = \sum_{r=1}^{R} f_{gr}\ e^{-t/t_{gr}} \tag{16}$$

$$\Phi_{gr}(t) = e^{-t/t_{gr}} \tag{17}$$

$\Phi_g(t)$ is the normalized additional concentration response function to an impulse of emission, and $\Phi_{gr}(t)$ are its components.

It follows from expression (15) that the additional concentration resulting from an impulse emission at time t = 0, of value e_{g0} , is:

$$\Delta r_g(t) = b_g\ e_{g0}\ \Phi_g(t) \tag{18}$$

The constant in the definition of the response function is such that $\Phi_g(0) = 1$. The normalized additional concentration response function to an impulse of emission $\Phi_g(t)$ is positive definite; it starts at one, decreases monotonically and tends asymptotically to zero at infinity.

The additional concentration resulting from constant emissions starting at t = 0, and of value $\bar{e}_g$ is:

$$\Delta r_g(t) = b_g\ \bar{e}_g \int_0^t \Phi_g(t-t')\, dt' = b_g\ \bar{t}_g\ \bar{e}_g\ \overline{\Phi}_g(t) = b_g\ \bar{t}_g\ \bar{e}_g \sum_{r=1}^{R} f_{gr}\ \overline{\Phi}_{gr}(t)$$

$$= b_g\ \bar{t}_g\ \bar{e}_g \sum_{r=1}^{R} f_{gr} \left[1 - (t_{gr}/\bar{t}_g)\, e^{-t/t_{gr}}\right] \tag{19}$$

$$\overline{\Phi}_{gr}(t) = 1 - (t_{gr}/\bar{t}_g)\, e^{-t/t_{gr}} \tag{20}$$

where $\overline{\Phi}_g(t)$ is the normalized additional concentration response function to constant emissions and $\overline{\Phi}_{gr}(t)$ are its components.

The constant in the definition of the response functions is such that $\lim_{t\to\infty} \overline{\Phi}_g(t) = 1$. The

normalized additional concentration response function to constant emissions $\overline{\Phi}_g(t)$ is positive definite; it starts at zero, increases monotonically and tends asymptotically to 1 at infinity.

From additional concentration to temperature increase

The relationship between additional concentration and temperature increase in expression (13) can be written as:

$$\Delta T_g(t) = \frac{(1/C)\,\overline{s}_g}{\overline{t}_c} \int_{-\infty}^{t} \Delta r_g(t')\,\Theta(t-t')\,dt' \qquad (21)$$

where:

$$\Theta(t) = \sum_{s=1}^{S} l_s\ \Theta_s(t) = \sum_{s=1}^{S} l_s\ (\overline{t}_c / t_{cs})\ e^{-t/t_{cs}} \qquad (22)$$

$$\Theta_s(t) = (\overline{t}_c / t_{cs})\ e^{-t/t_{cs}} \qquad (23)$$

$\Theta(t)$ is the normalized temperature increase response function to an impulse of additional concentration, and $\Theta_s(t)$ are its components.

It follows from expression (21) that the temperature increase resulting from an impulse of additional concentration at time t = 0, of value Δr_{g0}, is:

$$\Delta T_g(t) = \frac{(1/C)\,\overline{s}_g}{\overline{t}_c}\ \Delta r_{g0}\ \Theta(t) \qquad (24)$$

The constant in the definition of the response function is such that $\Theta(0) = 1$. The normalized temperature increase response function to an impulse of additional concentration is positive definite; it starts at one, decreases monotonically and tends asymptotically to zero at infinity.

The temperature increase resulting from constant additional concentration starting at t = 0, and of value $\Delta\overline{r}_g$ is:

$$\Delta T_g(t) = \frac{(1/C)\,\overline{s}_g}{\overline{t}_c}\ \Delta\overline{r}_g \int_0^t \Theta(t-t')\,dt' = (1/C)\,\overline{s}_g\ \Delta\overline{r}_g\ \overline{\Theta}(t) = (1/C)\,\overline{s}_g\ \Delta\overline{r}_g\ \sum_{s=1}^{S} l_s\ \overline{\Theta}_s(t)$$

$$= (1/C)\,\overline{s}_g\ \Delta\overline{r}_g\ \sum_{s-1}^{S} l_s\ (1 - e^{-t/t_{cs}}) \qquad (25)$$

$$\overline{\Theta}_s(t) = 1 - e^{-t/t_{cs}} \qquad (26)$$

where $\overline{\Theta}(t)$ is the normalized temperature increase response function to constant additional concentration and $\overline{\Theta}_s(t)$ are its components.

The constant in the definition of the response functions is such that $\lim_{t\to\infty} \overline{\Theta}(t) = 1$. The normalized temperature increase response function to constant additional concentration, $\overline{\Theta}(t)$ is positive definite; it starts at zero, increases monotonically and tends asymptotically to 1 at infinity.

Climate sensitivity

The asymptotic value of the temperature increase for a constant additional concentration of carbon dioxide starting at t = 0 and of value equal to the initial concentration is called the climate sensitivity. It is also described as the temperature increase for a doubling of the carbon dioxide concentration. It follows from (25) that:

$$cs = (1/C)\,\overline{s}_{CO_2}\, r_{CO_2 i} \tag{27}$$

and therefore

$$(1/C) = \frac{cs}{\overline{s}_{CO_2}\, r_{CO_2 i}} \tag{28}$$

where:

$r_{CO_2 i}$ is the initial carbon dioxide concentration that, as it is increased by the same amount, results in a temperature increase equal to the climate sensitivity.

From emissions to temperature increase

The relationship between emissions and temperature increase in expression (14) can be written as:

$$\Delta T_g(t) = (1/C)\,\overline{s}_g\; b_g\; \overline{t}_g \int_{-\infty}^{t} e_g(t')\,\Psi_g(t-t')\,dt' \tag{29}$$

where:

$$\Psi_g(t) = \sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr}\,\Psi_{grs}(t) = \sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr}\,\frac{(t_{gr}/\overline{t}_g)}{(t_{gr}-t_{cs})}\,(e^{-t/t_{gr}} - e^{-t/t_{cs}}) \tag{30}$$

$$\Psi_{grs}(t) = \frac{(t_{gr}/\overline{t}_g)}{(t_{gr}-t_{cs})}\,(e^{-t/t_{gr}} - e^{-t/t_{cs}}) \tag{31}$$

$\Psi_g(t)$ is the normalized temperature increase response function to an impulse of

emission, and $\Psi_{grs}(t)$ are its components.

For t_{gr} equal to t_{cs}, expression (31) contains the division of zero by zero. The limit in this case is:

$$\lim_{t_{cs}\to t_{gr}} \Psi_{grs}(t) = \frac{t}{(\bar{t}_g\ t_{cs})}\ e^{-t/t_{cs}} = \frac{t}{(\bar{t}_g\ t_{gr})}\ e^{-t/t_{gr}} \tag{32}$$

It follows from expression (29) that the temperature increase resulting from an impulse of emission at time t = 0, of value e_{g0}, is:

$$\Delta T_g(t) = (1/C)\ \bar{s}_g\ b_g\ \bar{t}_g\ e_{g0} \Psi_g(t) \tag{33}$$

The constant in the definition of the response function is such that $\int_0^\infty \Psi_g(t)\, dt = 1$.
The normalized temperature increase response function to an impulse of emission, $\Psi_g(t)$ is positive definite; it starts at zero, reaches a maximum and then tends asymptotically to zero at infinity.

The temperature increase resulting from constant emissions starting at t = 0, and of value $\bar{e}_g$, is:

$$\begin{aligned}\Delta \mathrm{T}_g(t) &= (1/C)\ \bar{s}_g\ b_g\ \bar{t}_g\ \bar{e}_g \int_0^t \Psi_g(t-t')\, dt' = (1/C)\ \bar{s}_g\ b_g\ \bar{t}_g\ \bar{e}_g\ \overline{\Psi}_g(t) \\ &= (1/C)\ \bar{s}_g\ b_g\ \bar{t}_g\ \bar{e}_g \sum_{s-1}^{S} l_s \sum_{r=1}^{R} f_{gr}\ \overline{\Psi}_{grs}(t) \\ &= (1/C)\ \bar{s}_g\ b_g\ \bar{t}_g\ \bar{e}_g \sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr} \left[1 - \frac{(t_{gr}/\bar{t}_g)}{(t_{gr} - t_{cs})} \left(t_{gr}\ e^{-t/t_{gr}} - t_{cs}\ e^{-t/t_{cs}}\right)\right]\end{aligned} \tag{34}$$

$$\overline{\Psi}_{grs}(t) = 1 - \frac{(t_{gr}/\bar{t}_g)}{(t_{gr} - t_{cs})} \left(t_{gr}\ e^{-t/t_{gr}} - t_{cs}\ e^{-t/t_{cs}}\right) \tag{35}$$

where $\overline{\Psi}_g(t)$ is the normalized temperature increase response function for constant emissions and $\overline{\Psi}_{grs}(t)$ are its components.

For t_{gr} equal to t_{cs}, expression (35) contains the division of zero by zero. The limit in this case is:

$$\lim_{t_{cs}\to t_{gr}} \overline{\Psi}_{grs}(t) = 1 - \frac{t + t_{gr}}{\bar{t}_g} e^{-t/t_{gr}} = 1 - \frac{t + t_{cs}}{\bar{t}_g} e^{-t/t_{cs}} \tag{36}$$

The constant in the definition of the response function is such that $\lim_{t\to\infty} \overline{\Psi}_g(t) = 1$. The normalized temperature increase response function to constant emissions, $\overline{\Psi}_g(t)$ is positive definite; it starts at zero, increases monotonically and tends asymptotically to 1 at infinity.

The temperature efficiency of a greenhouse gas

The constant factor in the expressions for the temperature increase as a function of emissions is defined as the temperature efficiency of a greenhouse gas, which can be written, with the help of expression (28), in terms of the climate sensitivity:

$$K_g = \frac{\overline{s}_g \; b_g \; cs \; \overline{t}_g}{\overline{s}_{CO_2} \; r_{CO_2 i}} \tag{37}$$

With this definition, the expressions for the additional concentration and temperature increase can be rewritten as:

$$\Delta T_g(t) = \frac{K_g}{b_g \; \overline{t}_g \; \overline{t}_c} \int_{-\infty}^{t} \Delta r_g(t') \; \Theta(t-t') \; dt' \tag{21'}$$

$$\Delta T_g(t) = \frac{K_g}{b_g \; \overline{t}_g \; \overline{t}_c} \; \Delta r_{g0} \; \Theta(t) \tag{24'}$$

$$\Delta T_g(t) = \frac{K_g}{b_g \; \overline{t}_g} \; \Delta \overline{r}_g \; \overline{\Theta}(t) \tag{25'}$$

$$\Delta T_g(t) = K_g \int_{-\infty}^{t} e_g(t') \; \Psi_g(t-t') \; dt' \tag{29'}$$

$$\Delta T_g(t) = K_g \; e_{g0} \; \Psi_g(t) \tag{33'}$$

$$\Delta \mathrm{T}_g(t) = K_g \; \overline{e}_g \; \overline{\Psi}_g(t) \tag{34'}$$

From emissions to temperature rate of change

The time rate of change of temperature is obtained by taking the derivative with respect to time of expression (30) and applying the result to expression (29'):

$$\frac{d \; \Delta T_g(t)}{d \; t} = K_g \sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr} \int_{-\infty}^{t} e_g(t') \frac{(t_{gr} / \overline{t}_g)}{(t_{gr} - t_{cs})} \left[(1/t_{cs}) \; e^{-(t-t')/t_{cs}} - (1/t_{gr}) \; e^{-(t-t')/t_{gr}} \right] dt' \tag{38}$$

The relationship between emissions and time rate of change of the temperature increase can be written as:

$$\frac{d \; \Delta T_g(t)}{d \; t} = \frac{K_g}{\overline{t}_g \; \overline{t}_c} \int_{-\infty}^{t} e_g(t') \; \Lambda_g(t-t') \; dt' \tag{39}$$

where:

$$\Lambda_g(t) = \sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr}\ \Lambda_{grs}(t) = \sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr}\ \frac{t_{gr}\ \bar{t}_c}{(t_{gr} - t_{cs})} \left[(1/t_{cs})\, e^{-t/t_{cs}} - (1/t_{gr})\, e^{-t/t_{gr}} \right] \tag{40}$$

$$\Lambda_{grs}(t) = \frac{t_{gr}\ \bar{t}_c}{(t_{gr} - t_{cs})} \left[(1/t_{cs})\, e^{-t/t_{cs}} - (1/t_{gr})\, e^{-t/t_{gr}} \right] \tag{41}$$

For t_{gr} equal to t_{cs} , expression (41) contains the division of zero by zero. The limit in this case is:

$$\lim_{t_{cs} \to t_{gr}} \Lambda_{grs}(t) = (\bar{t}_c / t_{cs})\,(1 - t/t_{cs})\, e^{-t/t_{cs}} = (\bar{t}_c / t_{gr})\,(1 - t/t_{gr})\, e^{-t/t_{gr}} \tag{42}$$

$\Lambda_g(t)$ is the normalized temperature rate of change response function to an impulse of emission, and $\Lambda_{grs}(t)$ are its components..

It follows from expression (39) that the temperature rate of change resulting from an impulse of emission at time t = 0, of value e_{g0} , is:

$$\frac{\Delta T_g(t)}{dt} = \frac{K_g}{\bar{t}_c\ \bar{t}_g}\, e_{g0}\ \Lambda_g(t) \tag{43}$$

The constant in the definition of the response function is such that $\Lambda_g(0) = 1$. The normalized temperature rate of change response function to an impulse of emission, $\Lambda_g(t)$ starts with the value one; it is initially positive, then negative and tends asymptotically to zero as time tends to infinity.

The temperature rate of change resulting from constant emissions starting at t = 0, and of value $\bar{e}_g$, is:

$$\frac{d\,\Delta T_g(t)}{d\,t} = \frac{K_g}{\bar{t}_c\ \bar{t}_g}\, \bar{e}_g \int_0^t \Lambda_g(t - t')\, dt' = K_g\ \bar{e}_g\ \bar{\Lambda}_g(t) = K_g\ \bar{e}_g \sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr}\ \bar{\Lambda}_{grs}(t)$$

$$= K_g\ \bar{e}_g \sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr}\ \frac{(t_{gr} / \bar{t}_g)}{(t_{gr} - t_{cs})} (e^{-t/t_{gr}} - e^{-t/t_{cs}}) \tag{44}$$

$$\bar{\Lambda}_{grs}(t) = \frac{(t_{gr} / \bar{t}_g)}{(t_{gr} - t_{cs})} (e^{-t/t_{gr}} - e^{-t/t_{cs}}) \tag{45}$$

where $\overline{\Lambda}_g(t)$ is the normalized temperature rate of change response function to constant emissions and $\overline{\Lambda}_{grs}(t)$ are its components.

This expression is the same as that for the normalized temperature increase response function to an impulse of emission $\left(\overline{\Lambda}_{grs}(t) = \Psi_{grs}(t)\right)$, which is to be expected since $\overline{\Lambda}_{grs}(t)$ results from taking the time integral and derivative of $\Psi_{grs}(t)$.

The constant in the definition of the response function is such that $\int_0^\infty \overline{\Lambda}_g(t)\, dt = 1$. The normalized temperature rate of change response function to constant emissions is positive definite; it starts at zero, reaches a maximum and then tends asymptotically to zero at infinity.

From emissions to mean sea level rise

The rise in mean sea level can be approximated by a multiple exponential response to a constant temperature increase starting at t = 0 :

$$\Delta msl_g(t) = \Delta\overline{T}_g \; MSL \; (1 - \sum_{m=1}^{M} h_m \; e^{-t/t_m}) \qquad (46)$$

where:

$\Delta msl_g(t)$ is the mean sea level rise resulting from a constant temperature increase in temperature starting at time t = 0 ;

$\Delta\overline{T}_g$ is the value of the constant temperature increase;

MSL is the asymptotic value of the mean sea level rise per unit of constant temperature increase;

h_m is the mth fraction of the mean sea level rise that adjusts exponentially with the time constant t_m ;

t_m is the exponential adjustment time constant of the fraction h_m.

It follows that the mean sea level rise response to an impulse of temperature increase of unit value is:

$$\Delta msl_g(t) = MSL \sum_{m=1}^{M} h_m \; (1/t_m)\, e^{-t/t_m} \qquad (47)$$

The time-dependent relationship between the temperature increase and mean sea level rise is then given by:

$$\Delta msl_g(t) = MSL \int_{-\infty}^{t} \Delta T(t') \sum_{m=1}^{M} h_m \, (1/t_m) \, e^{-(t-t')/t_m} \, dt' \tag{48}$$

Substitution of the expression for the temperature increase from (29') results in:

$$\Delta msl_g(t) = MSL \int_{-\infty}^{t} \left[K_g \int_{-\infty}^{t'} e_g(t'') \, \Psi_g(t'-t'') \, dt'' \right] \sum_{m=1}^{M} h_m \, (1/t_m) \, e^{-(t-t')/t_m} \, dt'$$

$$= MSL \; K_g \sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr} \sum_{m=1}^{M} h_m \frac{(t_{gr} / (\bar{t}_g \, t_m))}{(t_{gr} - t_{cs})} \int_{-\infty}^{t} \left[\int_{-\infty}^{t'} e_g(t'') \, (e^{-(t'-t'')/t_{gr}} - e^{-(t'-t'')/t_{cs}}) \, dt'' \right] e^{-(t-t')/t_m} \, dt' \tag{49}$$

The relationship between emissions and mean sea level rise in expression (49) can be written as:

$$\Delta msl_g(t) = MSL \; K_g \int_{-\infty}^{t} e(t') \, \Omega_g(t-t') \, dt' \tag{50}$$

where:

$$\Omega_g(t) = \sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr} \sum_{m=1}^{M} h_m \; \Omega_{grsm}(t) =$$

$$= \sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr} \sum_{m=1}^{M} h_m \frac{(t_{gr} / \bar{t}_g)}{(t_{gr} - t_{cs})} \left[\frac{t_{gr}}{(t_{gr} - t_m)} \left(e^{-t/t_{gr}} - e^{-t/t_m} \right) - \frac{t_{cs}}{(t_{cs} - t_m)} \left(e^{-t/t_{cs}} - e^{-t/t_m} \right) \right] \tag{51}$$

$$\Omega_{grsm}(t) = \frac{(t_{gr} / \bar{t}_g)}{(t_{gr} - t_{cs})} \left[\frac{t_{gr}}{(t_{gr} - t_m)} \left(e^{-t/t_{gr}} - e^{-t/t_m} \right) - \frac{t_{cs}}{(t_{cs} - t_m)} \left(e^{-t/t_{cs}} - e^{-t/t_m} \right) \right] \tag{52}$$

For two or three equal values of t_{gr} , t_{cs} and t_m, expression (52) contains the division of zero by zero. The limits in these cases are:

$$\lim_{t_{gr} \to t_m} \Omega_{grsm}(t) = \frac{(t_{gr} / \bar{t}_g)}{(t_{gr} - t_{cs})} \left[\frac{t}{t_{gr}} e^{-t/t_{gr}} - \frac{t_{cs}}{(t_{cs} - t_{gr})} \left(e^{-t/t_{cs}} - e^{-t/t_{gr}} \right) \right] =$$

$$= \frac{(t_m / \bar{t}_g)}{(t_m - t_{cs})} \left[\frac{t}{t_m} e^{-t/t_m} - \frac{t_{cs}}{(t_{cs} - t_m)} \left(e^{-t/t_{cs}} - e^{-t/t_m} \right) \right] \tag{53}$$

$$\lim_{t_{cs} \to t_m} \Omega_{grsm}(t) = \frac{(t_{gr} / \bar{t}_g)}{(t_{gr} - t_{cs})} \left[\frac{t_{gr}}{(t_{gr} - t_{cs})} \left(e^{-t/t_{gr}} - e^{-t/t_{cs}} \right) - \frac{t}{t_{cs}} e^{-t/t_{cs}} \right] =$$

$$= \frac{(t_{gr} / \bar{t}_g)}{(t_{gr} - t_m)} \left[\frac{t_{gr}}{(t_{gr} - t_m)} \left(e^{-t/t_{gr}} - e^{-t/t_m} \right) - \frac{t}{t_m} e^{-t/t_m} \right] \tag{54}$$

$$\lim_{t_{gr}\to t_{cs}} \Omega_{grsm}(t) = \frac{(t/\bar{t}_g)}{(t_{cs}-t_m)}\, e^{-t/t_{cs}} - \frac{(t_{cs}\, t_m/\bar{t}_g)}{(t_{cs}-t_m)^2}\left(e^{-t/t_{cs}} - e^{-t/t_m}\right) = \\ = \frac{(t/\bar{t}_g)}{(t_{gr}-t_m)}\, e^{-t/t_{gr}} - \frac{(t_{gr}\, t_m/\bar{t}_g)}{(t_{gr}-t_m)^2}\left(e^{-t/t_{gr}} - e^{-t/t_m}\right) \qquad (55)$$

$$\lim_{\substack{t_{gr}\to t_m \\ t_{cs}\to t_m}} \Omega_{grsm}(t) = (1/\bar{t}_g)\,(t^2/2t_{gr}^2)\, e^{-t/t_{gr}} = (1/\bar{t}_g)\,(t^2/2t_{cs}^2)\, e^{-t/t_{cs}} = (1/\bar{t}_g)\,(t^2/2t_m^2)\, e^{-t/t_m} \qquad (56)$$

$\Omega_g(t)$ is the normalized mean sea level rise response function to an impulse of emission, and $\Omega_{grsm}(t)$ are its components..

It follows from expression (50) that the mean sea level rise resulting from an impulse of emission at time t = 0, of value e_{g0} , is:

$$\Delta msl_g(t) = MSL\; K_g\; e_{g0}\; \Omega_g(t) \qquad (57)$$

The constant in the definition of the response function is such that $\int_0^\infty \Omega_g(t)\, dt = 1$. The normalized mean sea level rise response function to an impulse of emission, $\Omega_g(t)$ is positive definite; it starts with the value zero, is initially positive, then negative and tends asymptotically to zero as time tends to infinity.

The mean sea level rise resulting from constant emissions starting at t = 0, and of value $\bar{e}_g$, is:

$$\Delta msl_g(t) = MSL\; K_g\; \bar{e}_g \int_0^t \Omega_g(t-t')\, dt' = MSL\; K_g\; \bar{e}_g\; \overline{\Omega}_g(t) = MSL\; K_g\; \bar{e}_g \sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr} \sum_{m=1}^{M} h_m\; \overline{\Omega}_{grs}(t)$$

$$= MSL\; K_g\; \bar{e}_g \sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr} \sum h_m \left[\begin{array}{l} 1 - \dfrac{(t_{gr}/\bar{t}_g)\, t_{gr}^2}{(t_{gr}-t_m)(t_{gr}-t_{cs})}\, e^{-t/t_{gr}} \\ + \dfrac{(t_{gr}/\bar{t}_g)\, t_{cs}^2}{(t_{cs}-t_m)(t_{gr}-t_{cs})}\, e^{-t/t_{cs}} - \dfrac{(t_{gr}/\bar{t}_g)\, t_m^2}{(t_{gr}-t_m)(t_{cs}-t_m)}\, e^{-t/t_m} \end{array}\right] \qquad (58)$$

$$\overline{\Omega}_{grsm}(t) = \left[\begin{array}{l} 1 - \dfrac{(t_{gr}/\bar{t}_g)\, t_{gr}^2}{(t_{gr}-t_m)(t_{gr}-t_{cs})}\, e^{-t/t_{gr}} \\ + \dfrac{(t_{gr}/\bar{t}_g)\, t_{cs}^2}{(t_{cs}-t_m)(t_{gr}-t_{cs})}\, e^{-t/t_{cs}} - \dfrac{(t_{gr}/\bar{t}_g)\, t_m^2}{(t_{gr}-t_m)(t_{cs}-t_m)}\, e^{-t/t_m} \end{array}\right] \qquad (59)$$

where $\overline{\Omega}_g(t)$ is the normalized mean sea level rise response function to constant emissions and $\overline{\Omega}_{grsm}(t)$ are its components.

For two or three equal values of t_{gr} , t_{cs} and t_m, expression (59) contains the division of zero by zero. The limits in these cases are:

$$\lim_{t_{gr}\to t_m} \overline{\Omega}_{grsm}(t) = 1 + \frac{(t_m / \bar{t}_g)}{(t_{cs} - t_m)} \left[\left(2\,t_m + \frac{t_m^2}{(t_{cs} - t_m)} + t \right) e^{-t/t_m} - \frac{t_{cs}^2}{(t_{cs} - t_m)} e^{-t/t_{cs}} \right] =$$

$$= 1 + \frac{(t_{gr} / \bar{t}_g)}{(t_{cs} - t_{gr})} \left[\left(2\,t_{gr} + \frac{t_{gr}^2}{(t_{cs} - t_{gr})} + t \right) e^{-t/t_{gr}} - \frac{t_{cs}^2}{(t_{cs} - t_{gr})} e^{-t/t_{cs}} \right] \tag{60}$$

$$\lim_{t_{cs}\to t_m} \overline{\Omega}_{grsm}(t) = 1 + \frac{(t_{gr} / \bar{t}_g)}{(t_{gr} - t_m)} \left[\left(2\,t_m + \frac{t_m^2}{(t_{gr} - t_m)} + t \right) e^{-t/t_m} - \frac{t_{gr}^2}{(t_{gr} - t_m)} e^{-t/t_{gr}} \right] =$$

$$= 1 + \frac{(t_{gr} / \bar{t}_g)}{(t_{gr} - t_{cs})} \left[\left(2\,t_{cs} + \frac{t_m^2}{(t_{gr} - t_{cs})} + t \right) e^{-t/t_{cs}} - \frac{t_{gr}^2}{(t_{gr} - t_{cs})} e^{-t/t_{gr}} \right] \tag{61}$$

$$\lim_{t_{gr}\to t_{cs}} \overline{\Omega}_{grsm}(t) = 1 + \frac{(t_{gr} / \bar{t}_g)}{(t_m - t_{gr})} \left[\left(2\,t_{gr} + \frac{t_{gr}^2}{(t_m - t_{gr})} + t \right) e^{-t/t_{gr}} - \frac{t_m^2}{(t_m - t_{gr})} e^{-t/t_m} \right] =$$

$$= 1 + \frac{(t_{cs} / \bar{t}_g)}{(t_m - t_{cs})} \left[\left(2\,t_{cs} + \frac{t_{cs}^2}{(t_m - t_{cs})} + t \right) e^{-t/t_{cs}} - \frac{t_m^2}{(t_m - t_{cs})} e^{-t/t_m} \right] \tag{62}$$

$$\lim_{\substack{t_{gr}\to t_m \\ t_{cs}\to t_m}} \overline{\Omega}_{grsm}(t) = 1 - (t_m / \bar{t}_g) \left(1 + (t / t_m) + (t^2 / (2\,t_m^2)) \right) e^{-t/t_m} =$$

$$= 1 - (t_{gr} / \bar{t}_g) \left(1 + (t / t_{gr}) + (t^2 / (2\,t_{gr}^2)) \right) e^{-t/t_{gr}} = \tag{63}$$

$$= 1 - (t_{cs} / \bar{t}_g) \left(1 + (t / t_{cs}) + (t^2 / (2\,t_{cs}^2)) \right) e^{-t/t_{cs}}$$

The constant in the definition of the response function is such that $\lim_{t\to\infty} \overline{\Omega}_g(t) = 1$. The normalized mean sea level rise response function to constant emissions is positive definite; it starts at zero, increases monotonically and tends asymptotically to 1 at infinity.

4. Global warming potentials

A “carbon dioxide equivalent emission” is defined by means of a factor for each greenhouse gas other than carbon dioxide, such that their emissions may be added to those of carbon dioxide, after weighting by the respective factor.

The criterion used to choose the weighting factors is that the temperature increase after a specified time lag is the same as that which would be produced if there was a carbon dioxide emission equal in value to the carbon dioxide equivalent emission. Each

weighting factor is referred to as the global warming potential for greenhouse gas g. Thus, in general:

$$e_{CO_2\,equiv}(t) = e_{CO_2}(t) + \sum_g e_g(t)\,\Gamma_g \tag{64}$$

where

$\sum_g$ indicates a summation over the greenhouse gases other than carbon dioxide and

Γ_g is the global warming potential for greenhouse gas g, for a specified time lag.

In order to find the expression and time-dependence of the weighting factor, the temperature increase due to emissions of carbon dioxide and of other gases can be written from (29), with the definition of (64), as:

$$\Delta T(t) = K_{CO_2} \int_{-\infty}^{t} \left[e_{CO_2}(t') + \sum_g e_g(t')\,\Gamma_g(t-t') \right] \Psi_{CO_2}(t-t')\,dt' \tag{65}$$

where

$$\Gamma_g(t) = \frac{K_g\ \Psi_g(t)}{K_{CO_2}\ \Psi_{CO_2}(t)} \tag{66}$$

is the global warming potential for greenhouse gas g and time lag t.

The global warming potential can be written as a constant for each greenhouse gas, multiplied by a normalized global warming potential; after noting that $\lim_{t\to 0} \frac{\Psi_g(t)}{\Psi_{CO_2}(t)} = \frac{\bar{t}_{CO_2}}{\bar{t}_g}$ and requiring that $g_g(0) = 1$:

$$\Gamma_g(t) = \frac{\bar{s}_g\ b_g}{\bar{s}_{CO_2}\ b_{CO_2}}\, g_g(t) \tag{67}$$

$$g_g(t) = \frac{\sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr}\ \dfrac{t_{gr}}{(t_{gr} - t_{cs})}\,(e^{-t/t_{gr}} - e^{-t/t_{cs}})}{\sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{CO_2 r}\ \dfrac{t_{CO_2 r}}{(t_{CO_2 r} - t_{cs})}\,(e^{-t/t_{CO_2 r}} - e^{-t/t_{cs}})} \tag{68}$$

For impulse emissions at t = 0, of values $e_{CO_2 0}$ and e_{g0} , the resulting temperature increase can be written, from expression (33'), as:

$$\Delta T(t) = K_{CO_2} \left[e_{CO_2 0}(t) + \sum_g e_{g0}(t)\ \Gamma_g(t) \right] \Psi_{CO_2}(t) \tag{69}$$

For constant emissions starting at t = 0, of values $\bar{e}_{CO_2}$ and $\bar{e}_g$, the resulting temperature increase can be written, from expression (34'), as:

$$\Delta T(t) = K_{CO_2} \left[\bar{e}_{CO_2} + \bar{e}_g\ \overline{\Gamma}_g(t) \right] \overline{\Psi}_{CO_2}(t) \tag{70}$$

where:

$$\overline{\Gamma}_g(t) = \frac{K_g\ \overline{\Psi}_g(t)}{K_{CO_2}\ \overline{\Psi}_{CO_2}(t)} \tag{71}$$

is defined as the global warming potential commitment of greenhouse gas g and time lag t.

The global warming potential commitment can be written as a constant for each greenhouse gas, multiplied by a normalized global warming potential commitment, after noting that $\lim_{t \to 0} \frac{\overline{\Psi}_g(t)}{\overline{\Psi}_{CO_2}(t)} = \frac{\bar{t}_{CO_2}}{\bar{t}_g}$ and requiring that $\bar{g}_g(0) = 1$:

$$\overline{\Gamma}_g(t) = \frac{\bar{s}_g\ b_g\ \bar{t}_g}{\bar{s}_{CO_2}\ b_{CO_2}\ \bar{t}_{CO_2}}\ \bar{g}_g(t) \tag{72}$$

$$\bar{g}_g(t) = \frac{\sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr} \left[1 - \frac{(t_{gr} / \bar{t}_g)}{(t_{gr} - t_{cs})} \left(t_{gr}\ e^{-t/t_{gr}} - t_{cs}\ e^{-t/t_{cs}} \right) \right]}{\sum l_s \sum f_{CO_2 r} \left[1 - \frac{(t_{CO_2 r} / \bar{t}_{CO_2})}{(t_{CO_2 r} - t_{cs})} \left(t_{CO_2 r}\ e^{-t/t_{CO_2 r}} - t_{cs}\ e^{-t/t_{cs}} \right) \right]} \tag{73}$$

The IPCC GWP

The Intergovernmental Panel on Climate Change – IPCC defined GWP(t) as the ratio of the accumulated radiative forcing at time t, resulting from a unit impulse of additional concentration of greenhouse gas g at time t=0, and the accumulated radiative forcing at time t, resulting from a unit impulse of additional concentration of carbon dioxide at time t=0.

There is a fundamental difficulty with this definition, in that the accumulated radiative forcing is a variable that, once it reaches a certain value, it never returns to zero, even when the additional concentration returns to zero if all emissions are stopped.

The ratio adopted by the IPCC also corresponds to the ratio of temperature increases, under the same conditions, and with two additional limiting conditions: first, that all

additional concentration exponential decay time constants t_{gr} be very short in comparison with any of the temperature increase adjustment time constant t_{cs} ; and second, that the lag time t be much shorter than any t_{cs}. In addition, the definition of the IPCC $GWP_g(t)$ refers to a unit increase in additional concentration at time t=0, while the definition in this note refers to a unit impulse of emission, the difference between the two being the factor b .

The definition of the IPCC $GWP_g(t)$, in the notation used in this note, is:

$$GWP_g(t) = \frac{\int_0^t s_g \, \Delta r_g(t')\, dt}{\int_0^t s_{CO_2} \, \Delta r_{CO_2}(t')\, dt'} \tag{74}$$

It is to be noted that the IPCC uses the column value of the constant s , rather than the mean value $\bar{s}$ introduced in this note. To the extent that these constants appear only in the form of the ratio of the constant for a greenhouse gas to that for carbon dioxide, the difference is neglected in what follows.

Taking expression (2) for the additional concentration when $b_g = 1$ and for an impulse of concentration of value equal to one at time t = 0:

$$\Delta r_g(t) = \sum_{r=1}^{R} f_{gr} \, e^{-t/t_{gr}} \tag{75}$$

Substituting this value in expression (72):

$$GWP_g(t) = \frac{\int_0^t s_g \sum_{r=1}^{R} f_{gr} \, e^{-t'/t_{gr}} \, dt'}{\int_0^t s_{CO_2} \sum_{r=1}^{R} f_{CO_2 r} \, e^{-t'/t_{CO_2 r}} \, dt'}$$

$$= \frac{s_g \sum_{r=1}^{R} f_{gr} \, t_{gr} \, (1 - e^{-t/t_{gr}})}{s_{CO_2} \sum_{r=1}^{R} f_{CO_2 r} \, t_{CO_2 r} \, (1 - e^{-t/t_{CO_2 r}})} \tag{76}$$

The expression for the global warming potential as defined in this note is, from expressions (67) and (68):

$$\Gamma_g(t) = \frac{b_g \, s_g \sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr} \, \frac{t_{gr}}{(t_{gr} - t_{cs})} (e^{-t/t_{gr}} - e^{-t/t_{cs}})}{b_{CO_2} \, s_{CO_2} \sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{CO_2 r} \, \frac{t_{CO_2 r}}{(t_{CO_2 r} - t_{cs})} (e^{-t/t_{CO_2 r}} - e^{-t/t_{cs}})} \tag{77}$$

Considering the case when t << t_{cs} and $t_{gr} << t_{cs}$ for all values of s and r, and both b_g and b_{CO_2} are equal to one:

$$\Gamma_g(t) = \frac{s_g \sum_{r=1}^{R} f_{gr}\, t_{gr}\, (1 - e^{-t/t_{gr}})}{s_{CO_2} \sum_{r=1}^{R} f_{CO_2 r}\, t_{CO_2 r}\, (1 - e^{-t/t_{CO_2 r}})} \qquad (78)$$

which is the same as expression (75) for the IPCC GWP(t).

It follows that the IPCC GWP(t) is a special case of the global warming potential $\Gamma_g(t)$ defined in this note, for the case when b_g and b_{CO_2} are taken to be equal to 1, and the temperature increase adjustment time constant tends to infinity.

The "policy-maker model" of the Brazilian Proposal

The government of Brazil submitted to the Secretariat of the United Nations Framework Convention on Climate Change a proposal of elements of a protocol to that Convention in 1997. That proposal contained the suggestion of a "policy-maker" model as a simple means to translate emissions into temperature increase.

In the notation used in this note, the "policy-maker" model is:

$$\Delta T_g(t) = (1/C)\, s_g\, b_g \int_{-\infty}^{t} \left[\int_{-\infty}^{t'} e_g(t'')\, e^{-(t'-t'')/t_g}\, dt'' \right] dt' \qquad (79)$$

Inspection shows that this is the same as expression (14) in this note, with two approximations.

The temperature increase adjustment term is omitted in the Brazilian proposal, which is equivalent to considering the limit for the temperature increase adjustment time period tending to infinity. Such approximation is also made in the definition of the IPCC GWP(*t*).

The decay of the additional concentration is taken to follow a simple exponential law, that is, R is taken to be equal to one for all gases.

Even though the "policy-maker" model did not include the concept of a global warming potential, it is clear that it implies one such concept, which is similar to that of the IPCC GWP, with the addition of the constants b .

5. Non-linearities in the climate change response to emissions

There are certain non-linearities in the functional relationships between the emissions and the resulting climate response. Because these non-linearities affect the forcing of climate change, they are intrinsically different from the internal non-linearities in the

dynamics of the climate system. The latter are implicitly taken into account by the full atmosphere-ocean coupled general circulation models that are used for the derivation of the climate sensitivity.

The treatment of the non-linearities has two aspects to it. One is the estimation of the climate response to global emissions. The other is the response of the climate system to small changes in emissions from individual sources, this being the approach relevant to the attribution of cause to individual sources.

In this section, consideration is given to both the global and perturbation effect of the non-linearities associated with the non-linear dependence of the additional concentration upon emissions of carbon dioxide, and the non-linear dependence of the radiative forcing upon the additional concentration of carbon dioxide, methane and nitrous oxide.

Non-linear response of the additional concentration of carbon dioxide to emissions

The additional concentration of most greenhouse gases other than carbon dioxide can be well represented by a linear combination of the additional concentrations resulting from emissions by different sources. In the case of carbon dioxide this is not true for long periods of time, both due to the saturation of the carbon dioxide fertilization effect, and the saturation of the ocean surface waters.

The treatment of this non-linearity for global emissions can only be done with the use of a full carbon cycle model. For the purposes of this note, the "Bern" model (Joss et al, 1996)[4] is used. The "Bern" model was used in conjunction with a prescribed emissions scenario to compute the resulting additional concentration of carbon dioxide both for the prescribed emissions and the same with the superposition of a conveniently small pulse of emission, of magnitude 0.001GtC, at different points in time from 1770 to 2100.

For each starting time of the emission pulse, the resulting perturbation in atmospheric concentration was obtained by subtracting from the concentration resulting from the pulse perturbed emissions, that resulting from the prescribed unperturbed emissions.

In each case the perturbation in atmospheric concentration was expressed as a linear combination of exponential functions, with the same 10 characteristic exponential time constants used in the Bern model, and coefficients determined by a least-square technique. It was found that this representation does not depart from the results of the calculation by more than 3% in the case of a pulse in 1770, and not more than .2% for a pulse in 1990.

The application of this result into (14) results in the following expression (where it is to be noted that the coefficients f_{gr} are now a function of t"):

[4] The Fortran code of the HILDA, or Bern model, was kindly supplied by Prof. Fortunat Joos

$$\Delta T_g(t) = (1/C)\,\bar{s}_g\;b_g\int_{-\infty}^{t}\left\{\int_{-\infty}^{t'} e_g(t'')\left[\sum_{r=1}^{R} f_{gr}(t'')\;e^{-(t'-t'')/t_{gr}}\right]dt''\right\}\left[\sum_{s=1}^{S} l_s\;(1/t_{cs})\,e^{-(t-t')/t_{cs}}\right]dt' \tag{80}$$

This relationship between emissions and temperature increase can be written in a form similar, but not equal, to that of expressions (29) to (31). Substitution of (30) into (29) together with the recognition that f_{gr} is a function of time results in:

$$\Delta T_g(t) = (1/C)\,\bar{s}_g\;b_g\;\bar{t}_{gr0}\int_{-\infty}^{t} e_g(t')\sum_{s=1}^{S} l_s\sum_{r=1}^{R} f_{gr}(t')\frac{(t_{gr}/\bar{t}_{gr0})}{(t_{gr}-t_{cs})}(e^{-t/t_{gr}}-e^{-t/t_{cs}})\,dt' \tag{81}$$

where

$\bar{t}_{gr0}$ is the effective time constant computed with the values $f_{gr}(t_0)$.

Individual components of the temperature increase can be defined by means of

$$\Delta T_g(t)=\sum_{r=1}^{R} f_{gr}(t_0)\,\Delta T_{gr}(t) \tag{82}$$

Then,

$$\Delta T_{gr}(t) = (1/C)\,\bar{s}_g\;b_g\,\bar{t}_{gr0}\int_{-\infty}^{t} e_g(t')\frac{f_{gr}(t')}{f_{gr}(t_0)}\sum_{s=1}^{S} l_s\frac{(t_{gr}/\bar{t}_{gr0})}{(t_{gr}-t_{cs})}(e^{-t/t_{gr}}-e^{-t/t_{cs}})dt' \tag{83}$$

A component normalized temperature increase response function to an impulse of emission can be defined as:

$$y_{gr}(t) = \sum_{s=1}^{S} l_s\frac{(t_{gr}/\bar{t}_{gr0})}{(t_{gr}-t_{cs})}(e^{-t/t_{gr}}-e^{-t/t_{cs}}) \tag{84}$$

so that an individual component of the temperature increase can be written as:

$$\Delta T_{gr}(t)=(1/C)\,\bar{s}_g\;b_g\,\bar{t}_{gr0}\int_{-\infty}^{t} e_g(t')\left[1+\frac{(f_{gr}(t')-f_{gr}(t_0))}{f_{gr}(t_0)}\right]y_{gr}(t-t')\,dt' \tag{85}$$

The term is square brackets can be interpreted as either a correction to the emission pulse at time t' or, alternatively, as a factor, dependent upon t', that affects the component normalized temperature increase response function. If wished, it can be written as a power series in the variable $(t-t_0)$:

$$\Delta T_{gr}(t) = (1/C)\,\bar{s}_g\; b_g \int_{-\infty}^{t} e_g(t') \left[1 + \sum_{m=1}^{M_{gr}} g_{grm} (t - t_0)^m \right] y_{gr}(t - t')\, dt' \qquad (86)$$

with the coefficients g_{grm} determined by a least-square technique from the results of a perturbation run of a carbon-cycle model. This expression is only valid within the period for which the coefficients were determined.

Non-linear response of the mean radiative forcing to additional concentration

The mean radiative forcing $\bar{s}_g$ is actually not constant, but rather it is a function of the atmospheric concentration, for carbon dioxide, methane and nitrous oxide. Expression (7) should then be modified to:

$$\Delta \bar{Q}_g(t) = \bar{s}_g(\Delta r_g(t))\, \Delta r_g(t) \qquad (87)$$

Substitution into (10) results in a modified expression (13), which can be written in terms of the temperature increase response function to an impulse of additional concentration:

$$\Delta T_g(t) = (1/C) \int_{-\infty}^{t} \bar{s}_g(\Delta r_g^G(t'))\, \Delta r_g^G(t') \left[\sum_{s=1}^{S} l_s\, (1/t_{cs})\, e^{-(t-t')/t_{cs}} \right] dt' \qquad (88)$$

or, using the definition of the normalized temperature increase response function to an impulse of additional concentration from (22),

$$\Delta T_g(t) = \frac{cs}{\bar{t}_c\; \bar{s}_{CO_2}\; r_{CO_2 i}} \int_{-\infty}^{t} \bar{s}_g\, (\Delta r_g^G(t'))\, \Delta r_g^G(t')\, \Theta(t - t')\, dt' \qquad (89)$$

where the superscript G refers to global additional concentrations.

Combination with expression (2) provides the expression for the relationship between global emissions of greenhouse gas g and the resulting temperature increase, written with use of the temperature increase response function to an impulse of additional concentration and the additional concentration response function to an impulse of emission:

$$\Delta T_g(t) = (1/C)\; b_g \int_{-\infty}^{t} \bar{s}_g \left(\int_{-\infty}^{t'} e_g^G(t'')\, \Phi_g(t''-t') \right) \left\{ \int_{-\infty}^{t'} e_g(t'')\, \Phi_g(t''-''t')\, dt'' \right\} \Theta(t - t')\, dt' \qquad (90)$$

This formula can only be used with numerical integration, because the non-linear dependence of the radiative forcing upon the atmospheric concentration of carbon dioxide, methane and nitrous oxide are such that an analytical solution can not be found.

In the special case of constant emissions, expression (89) is simplified and the asymptotic limit of the temperature increase as time tends to infinity can be written as:

$$\lim_{t\to\infty} \Delta\bar{T}_g^{\,G}(t) = \frac{\bar{s}_g(\lim_{t\to\infty} \Delta\bar{r}_g^{\,G}(t))\ c.s.}{\bar{s}_{CO_2}(r_{CO_2 i})} = \frac{\bar{s}_g(b_g\ \bar{t}_g\ \bar{e}_g^{\,G})\ c.s.}{\bar{s}_{CO_2}(r_{CO_2 i})} \qquad (91)$$

Non-linear attribution of climate change for prescribed additional concentrations

When using the response functions to estimate the relative effect of emissions from different sources, prescribed atmospheric concentrations can be used to determine the appropriate mean radiative forcing. An analytical expression for the response functions can be found if the time dependence of the mean radiative forcing is expressed as a power series, truncated to provide the desired accuracy.

Given atmospheric concentration data for a certain period of time $r_g(t)$, the mean radiative forcing can be written as:

$$\bar{s}_g(t) = \bar{s}_g(r_g(t)) = \bar{s}_{g0}\left[1 + \sum_{n=1}^{N_g} a_{gn}\,(t - t_0)^n\right] \qquad (92)$$

where:

$\bar{s}_{g0}$ is the mean radiative forcing at time t_0 ;

N_g is the order of the expansion;

a_{gn} are coefficients determined from the data by a least square technique;

Substitution of expression (91) into the full expression (14) for the temperature increase allows the determination of the normalized temperature increase response function taking into account the non-linearity in the relationship between additional concentration and mean radiative forcing:

$$\Psi_{grs}(t) = \frac{(t_{gr}\ /\ \bar{t}_g)}{(t_{gr} - t_{cs})}\left\{\begin{array}{l} e^{-t/t_{gr}}\left[1 + \sum_{n=1}^{N_g} a_{gn} \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!} \frac{(t-t_0)^{n-k}}{((t_{gr} - t_{cs})/(t_{gr} t_{cs}))^k}\right] - \\ - e^{-t/t_{cs}}\left[1 + \sum_{n=1}^{N_g} a_{gn} \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!} \frac{(-t_0)^{n-k}}{((t_{gr} - t_{cs})/(t_{gr} t_{cs}))^k}\right] \end{array}\right\}$$

(93)

This expression is only valid within the time period for which the coefficients a_{gn} were obtained.

6. The effect of emissions over specified time periods

The separation of the effects of emissions occurring over different time periods can be obtained by separating the time integrals into a sum of integrals over each time interval previous to the time of interest.

For the sake of simplicity in the notation, the variables and functions in this section are written in terms of their s and r components. The full expressions are then obtained by summing over the components after weighting with the factors l_s and f_{gr}, as appropriate.

Care should be taken, however, that the summation over the components can only be made for the full expression. There are products in the expressions, and the addition and multiplication operations cannot be interchanged.

The following notation is introduced for the additional concentration and temperature increase components, respectively, at the end of the time period $(t_a\ ,\ t_b)$, resulting from emissions during that time period.

$$\Delta r_{gr}(t_a\ ,t_b) = b_g \int_{t_a}^{t_b} e_g(t')\ \Phi_{gr}(t_b - t')\ dt' \tag{94}$$

and

$$\Delta T_{grs}(t_a\ ,\ t_b) = K_g \int_{t_a}^{t_b} e_g(t')\ \Psi_{grs}(t_b - t')\ dt' \tag{95}$$

Emissions over several periods

The time before t is divided into n+1 intervals, $(-\infty\ ,\ t_0)$, $(t_0\ ,\ t_1)$, …, $(t_{n-1}\ ,\ t_n)$, $(t_n\ ,\ t)$. The relationship between emissions and the additional concentration components can then be written as:

$$\Delta r_{gr}(t) = \Delta r_{gr}\ (-\infty\ ,\ t) = b_g \int_{-\infty}^{t} e_g(t')\ \Phi_{gr}(t-t')\ dt' =$$

$$= b_g \int_{-\infty}^{t_0} e_g(t')\ \Phi_{gr}(t-t')\ dt' + b_g \int_{t_0}^{t_1} e_g(t')\ \Phi_{gr}(t-t')\ dt' + \ldots\ldots +$$

$$+ b_g \int_{t_{n-1}}^{t_n} e_g(t')\ \Phi_{gr}(t-t')\ dt' + b_g \int_{t_n}^{t} e_g(t')\ \Phi_{gr}(t-t')\ dt'$$

$$= b_g \sum_{i=0}^{n} \int_{t_{i-1}}^{t_i} e_g(t')\ \Phi_{gr}(t-t')\ dt' + b_g \int_{t_n}^{t} e_g(t')\ \Phi_{gr}(t-t')\ dt' \tag{96}$$

where it is understood that t_{-1} represents t tending to minus infinity.

This expression contains integrals of the following type, which can be rewritten as shown:

$$\int_{t_a}^{t_b} e(t')\ e^{-(t-t')/t}\ dt' = e^{-t/t} \int_{t_a}^{t_b} e(t')\ e^{t'/t}\ dt' = e^{-(t-t_b)/t} \int_{t_a}^{t_b} e(t')\ e^{-(t_b-t')/t}\ dt' \tag{97}$$

The use of this equality allows the additional concentration component to be written as:

$$\Delta r_{gr}(t) = b_g \sum_{i=0}^{n} e^{-(t-t_i)/t_{gr}} \int_{t_{i-1}}^{t_i} e_g(t')\, e^{-(t_i-t')/t_{gr}}\, dt' + b_g \int_{t_n}^{t} e_g(t')\, e^{-(t-t')/t_{gr}}\, dt' \qquad (98)$$

The use of the definition of the normalized response function from expression (17) allows the additional concentration component to be written in the following two equivalent forms:

$$\Delta r_{gr}(t) = \sum_{i=0}^{n} \Delta r_{gr}(t_{i-1}\,,\, t_i)\, \Phi_{gr}(t - t_i) + \Delta r_{gr}(t_n\,,\, t) \qquad (99)$$

$$\Delta r_{gr}(t) = \left[\sum_{i=0}^{n} \Delta r_{gr}(t_{i-1}\,,\, t_i)\, \Phi_{gr}(t_n - t_i) \right] \Phi_{gr}(t - t_n) + \Delta r_{gr}(t_n\,,\, t) \qquad (100)$$

It is possible to write the full expression for the additional concentration, by defining modified weighting factors f'_{gr}, as follows:

$$\Delta r_g(t) = \Delta r_g(t_n) \sum_{r=1}^{R} f'_{gr}\, \Phi_{gr}(t - t_n) + \Delta r_g(t_n\,,\, t) \qquad (101)$$

where:

$$f'_{gr} = f_{gr}\, \frac{\sum_{i=0}^{n} \Delta r_{gr}(t_{i-1}\,,\, t_i)\, \Phi_{gr}(t_n - t_i)}{\Delta r_g(t_n)} \qquad (102)$$

A similar development can be made starting with expression (97) for the temperature increase component as a function of emissions:

$$\Delta T_{grs}(t) = \Delta T_{grs}(-\infty\,,\, t) = K_g \int_{-\infty}^{t} e_g(t')\, \Psi_{grs}(t - t')\, dt'$$

$$= K_g \sum_{i=0}^{n} \int_{t_{i-1}}^{t_i} e_g(t')\, \Psi_{grs}(t - t')\, dt' + K_g \int_{t_n}^{t} e_g(t')\, \Psi_{grs}(t - t')\, dt' \qquad (103)$$

$$\Delta T_{grs}(t) = \frac{K_g\, (t_{gr} / \bar{t}_g)}{(t_{gr} - t_{cs})} \left\{ \begin{array}{l} \sum_{i=0}^{n} \left[e^{-(t-t_i)/t_{gr}} \int_{t_{i-1}}^{t_i} e_g(t')\, e^{-(t_i-t')/t_{gr}}\, dt' - e^{-(t-t_i)/t_{cs}} \int_{t_{i-1}}^{t_i} e_g(t')\, e^{-(t_i-t')/t_{cs}}\, dt' \right] + \\ + \int_{t_n}^{t} e_g(t') \left(e^{-(t-t')/t_{gr}} - e^{-(t-t')/t_{cs}} \right) dt' \end{array} \right\} \qquad (104)$$

The above expression can be rewritten by subtracting and adding to the first line the integral in the left multiplied by the exponential factor with the constant t_{cs} and regrouping:

$$\Delta T_{grs}(t) = \frac{K_g\ (t_{gr}/\bar{t}_g)}{(t_{gr} - t_{cs})} \left\{ \sum_{i=0}^{n} \left[\begin{array}{l} e^{-(t-t_i)/t_{gr}} \int_{t_{i-1}}^{t_i} e_g(t')\, e^{-(t_i-t')/t_{gr}}\, dt' - e^{-(t-t_i)/t_{cs}} \int_{t_{i-1}}^{t_i} e_g(t')\, e^{-(t_i-t')/t_{gr}}\, dt' - \\ - e^{-(t-t_i)/t_{cs}} \int_{t_{i-1}}^{t_i} e_g(t')\, e^{-(t_i-t')/t_{cs}}\, dt' + e^{-(t-t_i)/t_{cs}} \int_{t_{i-1}}^{t_i} e_g(t')\, e^{-(t_i-t')/t_{gr}}\, dt' \end{array} \right] + \right.$$
$$\left. + \int_{t_n}^{t} e_g(t') \left(e^{-(t-t')/t_{gr}} - e^{-(t-t')/t_{cs}} \right) dt' \right\}$$

$$= \frac{K_g\ (t_{gr}/\bar{t}_g)}{(t_{gr} - t_{cs})} \left\{ \sum_{i=0}^{n} \left[\begin{array}{l} \left(e^{-(t-t_i)/t_{gr}} - e^{-(t-t_i)/t_{cs}} \right) \int_{t_{i-1}}^{t_i} e_g(t')\, e^{-(t_i-t')/t_{gr}}\, dt' + \\ + e^{-(t-t_i)/t_{cs}} \int_{t_{i-1}}^{t_i} e_g(t') \left(e^{-(t_i-t')/t_{gr}} - e^{-(t_i-t')/t_{cs}} \right) dt' \end{array} \right] + \right.$$
$$\left. + \int_{t_n}^{t} e_g(t') \left(e^{-(t-t')/t_{gr}} - e^{-(t-t')/t_{cs}} \right) dt' \right\} \quad (105$$

The use of the definition of the normalized response functions allows the temperature increase component to be written in the following two equivalent forms:

$$\begin{aligned} \Delta T_{grs}(t) = {} & (t_{cs}/\bar{t}_c) \sum_{i=0}^{n} \Delta T_{grs}(t_{i-1},\, t_i)\, \Theta_s(t - t_i) + \\ & + \frac{K_g}{b_g} \sum_{i=0}^{n} \Delta r_{gr}(t_{i-1}, t_i)\, \Psi_{grs}(t - t_i) + \\ & + \Delta T_{grs}(t_n\ , t) \end{aligned} \quad (106)$$

or

$$\begin{aligned} \Delta T_{grs}(t) = {} & (t_{cs}/\bar{t}_c) \left[\sum_{i=0}^{n} \left((t_{cs}/\bar{t}_c)\, \Delta T_{grs}(t_{i-1}, t_i)\, \Theta_s(t_n - t_i) + \frac{K_g}{b_g}\, \Delta r_{gr}(t_{i-1}, t_i)\, \Psi_{grs}(t_n - t_i) \right) \right] \Theta_s(t - t_n) + \\ & + \frac{K_g}{b_g} \left[\sum_{i=0}^{n} \Delta r_{gr}(t_{i-1}, t_i)\, \Phi_{gr}(t_n - t_i) \right] \Psi_{grs}(t - t_n) + \\ & + \Delta T_{grs}(t_n\ , t) \end{aligned} \quad (107)$$

It is possible to write the full expression for the temperature increase, by using the modified weighting factors f'_{gr} and defining a modified weighting factor l'_s, as follows:

$$\Delta T_g(t) = \Delta T_g(t_n) \sum_{s=1}^{S} l_s^{'}\ \Theta_s(t - t_n) +$$
$$+ \frac{K_g}{b_g} \Delta r_g(t_n) \sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr}^{'}\ \Psi_{grs}(t - t_n) + \qquad (108)$$
$$+ \Delta T_g(t_n\ , t)$$

where:

$$l_s^{'} = l_s \frac{(t_{cs}/\bar{t}_c) \sum_{r=1}^{R} f_{gr} \sum_{i=0}^{n} \left[(t_{cs}/\bar{t}_c) \Delta T_{grs}(t_{i-1}\ , t_i)\ \Theta_s(t_n - t_i) + \frac{K_g}{b_g} \Delta r_{gr}(t_{i-1}, t_i)\ \Psi_{grs}(t_n - t_i) \right]}{\Delta T_g(t_n)} \qquad (109)$$

Emissions over one period and afterwards

For emissions occurring during the period (t_a , t_b) and afterwards, the additional concentration component, from (100), is simplified to:

$$\Delta r_{gr}(t) = \Delta r_{gr}(t_a\ , t_b)\ \Phi_{gr}(t - t_b) + \Delta r_{gr}(t_b\ , t) \qquad (110)$$

Comparison of expressions (18) and (110) shows that the additional concentration after the period of emissions is equal to that resulting from an impulse of emission of value $\Delta r_g(t_a\ , t_b) / b_g$ at time $t = t_b$, which then decays with time according to the normalized additional concentration response function to an impulse of emission $\Phi_{gr}(t)$.

In the general case of emissions occurring in different time periods, inspection of expressions (101) and (102) shows that the situation is similar. The additional concentration at the end of each period, t_i , is equal to that resulting from an impulse of emission of value $\Delta r_{gr}(t_{i-1}\ , t_i) / b_g$ at time $t = t_i$, which then decays according to the normalized additional concentration response function to an impulse of emission $\Phi_{gr}(t)$.

The full expressions for the additional concentration and the modified weighting factors f'_{gr} , from (101) and (102), become:

$$\Delta r_g(t) = \Delta r_g(t_b) \sum_{r=1}^{R} f_{gr}^{'}\ \Phi_{gr}(t - t_b) + \Delta r_g(t_b\ , t) \qquad (111)$$

where:

$$f_{gr}^{'} = f_{gr} \frac{\Delta r_{gr}(t_a\ , t_b)}{\Delta r_g(t_b)} \qquad (112)$$

Similarly, for emissions occurring during the period (t_a , t_b) and afterwards, the temperature increase component, from (106), is simplified to:

$$\Delta T_{grs}(t) = (t_{cs} / \bar{t}_c)\, \Delta T_{grs}(t_a\ , t_b)\, \Theta_s(t - t_b) + \frac{K_g}{b_g}\, \Delta r_{grs}(t_a\ , t_b)\, \Psi_{grs}(t - t_b) + \Delta T_g(t_b\ , t)$$

(113)

Comparison of expressions (24) and (33) with the first and second terms of (113), respectively, shows that the temperature increase after the period of emissions is partly equal to that resulting from the temperature increase at the end of the emissions period, t_b, after decaying according to the temperature increase response function to an impulse of additional concentration, $\Theta_s(t)$; and partly equal to that resulting from an impulse of emission of value $\Delta r_{gr}(t_a\ , t_b) / b_g$ at time $t = t_b$.

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The full expressions for the temperature increase and the modified weighting factors $l_s^{'}$, from (108) and (109) become:

$$\begin{aligned} \Delta T_g(t) = {} & \Delta T_g(t_b) \sum_{s=1}^{S} l_s^{'}\, \Theta_s(t - t_b) + \\ & + \frac{K_g}{b_g}\, \Delta r_g(t_b) \sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr}^{'}\, \Psi_{grs}(t - t_b) + \\ & + \Delta T_g(t_b\ , t) \end{aligned} \qquad (114)$$

where:

$$l_s^{'} = l_s\, \frac{(t_{cs} / \bar{t}_c) \sum_{r=1}^{R} f_{gr}\, \Delta T_{grs}(t_a\ , t_b)}{\Delta T_g(t_b)} \qquad (115)$$

7. Summary of formulas

$$(1/C) = \frac{cs}{\bar{s}_{CO_2}\, r_{CO_2 i}} \qquad (28)$$

$$\mathrm{K}_g = \frac{\bar{s}_g\ b_g\ cs\ \bar{t}_g}{\bar{s}_{CO_2}\ r_{CO_2 i}} \qquad (36)$$

<u>Response functions to impulses</u>

Additional concentration response function to an impulse of emission

$$\Delta r_g(t) = b_g \int_{-\infty}^{t} e_g(t')\, \Phi_g(t - t')\, dt' \qquad (15)$$

$$\Phi_g(t) = \sum_{r=1}^{R} f_{gr}\ e^{-t/t_{gr}} \tag{16}$$

Temperature increase response function to an impulse of additional concentration

$$\Delta T_g(t) = (cs / r_{CO_2 i})\ (\bar{s}_g / \bar{s}_{CO_2})\ (1/\bar{t}_c) \int_{-\infty}^{t} \Delta r_g(t')\ \Theta(t-t')\ dt' \tag{21}$$

$$\Theta(t) = \sum_{s=1}^{S} l_s\ (\bar{t}_c / t_{cs})\ e^{-t/t_{cs}} \tag{22}$$

Temperature increase response function to an impulse of emission

$$\Delta T_g(t) = (cs / r_{CO_2 i})\ (\bar{s}_g / \bar{s}_{CO_2})\ b_g\ \bar{t}_g \int_{-\infty}^{t} e_g(t')\ \Psi_g(t-t')\ dt' \tag{29}$$

$$\Psi_g(t) = \sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr}\ \frac{(t_{gr} / \bar{t}_g)}{(t_{gr} - t_{cs})} (e^{-t/t_{gr}} - e^{-t/t_{cs}}) \tag{31}$$

Temperature rate of change response function to an impulse of emission

$$\frac{d\,\Delta T_g(t)}{d\,t} = (cs / r_{CO_2 i})\ (\bar{s}_g / \bar{s}_{CO_2})\ b_g\ (1/\bar{t}_c) \int_{-\infty}^{t} e_g(t')\ \Lambda_g(t-t')\ dt' \tag{39}$$

$$\Lambda_g(t) = \sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr}\ \frac{t_{gr}\ \bar{t}_c}{(t_{gr} - t_{cs})} \left[(1/t_{cs})\ e^{-t/t_{cs}} - (1/t_{gr})\ e^{-t/t_{gr}}\right] \tag{40}$$

Mean sea-level rise response function to an impulse of emission

$$\Delta msl_g(t) = MSL\ K_g \int_{-\infty}^{t} e(t')\ \Omega_g(t-t')\ dt' \tag{50}$$

$$\Omega_g(t) = \sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr} \sum_{m=1}^{M} h_m\ \frac{(t_{gr} / \bar{t}_g)}{(t_{gr} - t_{cs})} \left[\frac{t_{gr}}{(t_{gr} - t_m)} \left(e^{-t/t_{gr}} - e^{-t/t_m}\right) - \frac{t_{cs}}{(t_{cs} - t_m)} \left(e^{-t/t_{cs}} - e^{-t/t_m}\right)\right] \tag{51}$$

<u>Response functions to constant values</u>

Additional concentration response function to constant emissions

$$\Delta r_g(t) = b_g\ \bar{t}_g\ \bar{e}_g\ \overline{\Phi}_g(t) \tag{19}$$

$$\overline{\Phi}_g(t) = \sum_{r=1}^{R} f_{gr}\ (t_{gr} / \bar{t}_g)\ (1 - e^{-t/t_{gr}}) \tag{20}$$

Temperature increase response function to constant additional concentration

$$\Delta T_g(t) = (cs / r_{CO_2 i})\ (\bar{s}_g / \bar{s}_{CO_2})\ \Delta \bar{r}_g\ \overline{\Theta}(t) \tag{25}$$

$$\overline{\Theta}(t) = \sum_{s=1}^{S} l_s\ (1 - e^{-t/t_{cs}}) \tag{26}$$

Temperature increase response function to constant emissions

$$\Delta T_g(t) = (cs / r_{CO_2 i})\ (\bar{s}_g / \bar{s}_{CO_2})\ b_g\ \bar{t}_g\ \bar{e}_g\ \overline{\Psi}_g(t) \tag{34}$$

$$\overline{\Psi}_g(t) = \sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr} \left[1 - \frac{(t_{gr} / \bar{t}_g)}{(t_{gr} - t_{cs})} \left(t_{gr}\, e^{-t/t_{gr}} - t_{cs}\, e^{-t/t_{cs}} \right) \right] \tag{35}$$

Temperature rate of change response function to constant emissions

$$\frac{d\,\Delta T_g(t)}{d\,t} = (cs / r_{CO_2 i})\,(\bar{s}_g / \bar{s}_{CO_2})\, b_g\, \bar{t}_g\, \bar{e}_g\, \overline{\Lambda}_g(t) \tag{44}$$

$$\overline{\Lambda}_g(t) = \sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr}\, \frac{(t_{gr} / \bar{t}_g)}{(t_{gr} - t_{cs})} (e^{-t/t_{gr}} - e^{-t/t_{cs}}) \tag{45}$$

Mean sea-level rise response function to constant emissions

$$\Delta msl_g(t) = MSL\, K_g\, \bar{e}_g\, \overline{\Omega}_g(t) \tag{58}$$

$$\Delta msl_g(t) = MSL\; K_g\, \bar{e}_g \sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr} \sum h_m \left[1 - \frac{(t_{gr} / \bar{t}_g)\, t_{gr}^2}{(t_{gr} - t_m)(t_{gr} - t_{cs})} e^{-t/t_{gr}} + \frac{(t_{gr} / \bar{t}_g)\, t_{cs}^2}{(t_{cs} - t_m)(t_{gr} - t_{cs})} e^{-t/t_{cs}} - \frac{(t_{gr} / \bar{t}_g)\, t_m^2}{(t_{gr} - t_m)(t_{cs} - t_m)} e^{-t} \right. \tag{59}$$

Global warming potentials

Global warming potential

$$\Delta T(t) = K_{CO_2} \int_{-\infty}^{t} \left[e_{CO_2}(t') + \sum_g e_g(t')\, \Gamma_g(t - t') \right] \Psi_{CO_2}(t - t')\, dt' \tag{65}$$

$$\Gamma_g(t) = (\bar{s}_g / \bar{s}_{CO_2})\,(b_g / b_{CO_2})\, g_g(t) \tag{67}$$

$$g_g(t) = \frac{\sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr}\, \frac{t_{gr}}{(t_{gr} - t_{cs})} (e^{-t/t_{gr}} - e^{-t/t_{cs}})}{\sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{CO_2 r}\, \frac{t_{CO_2 r}}{(t_{CO_2 r} - t_{cs})} (e^{-t/t_{CO_2 r}} - e^{-t/t_{cs}})} \tag{68}$$

Global warming potential commitment

$$\Delta T(t) = K_{CO_2} \left[\bar{e}_{CO_2} + \bar{e}_g\, \overline{\Gamma}_g(t) \right] \overline{\Psi}_{CO_2}(t) \tag{70}$$

$$\overline{\Gamma}_g(t) = (\bar{s}_g / \bar{s}_{CO_2})\,(b_g / b_{CO_2})\,(\bar{t}_g / \bar{t}_{CO_2})\, \bar{g}_g(t) \tag{72}$$

$$\bar{g}_g(t) = \frac{\sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr} \left[1 - \frac{(t_{gr} / \bar{t}_g)}{(t_{gr} - t_{cs})} \left(t_{gr}\, e^{-t/t_{gr}} - t_{cs}\, e^{-t/t_{cs}} \right) \right]}{\sum l_s \sum f_{CO_2 r} \left[1 - \frac{(t_{CO_2 r} / \bar{t}_{CO_2})}{(t_{CO_2 r} - t_{cs})} \left(t_{CO_2 r}\, e^{-t/t_{CO_2 r}} - t_{cs}\, e^{-t/t_{cs}} \right) \right]} \tag{73}$$

IPCC GWP

$$GWP_g(t) = \frac{s_g \sum_{r=1}^{R} f_{gr}\, t_{gr}\, (1 - e^{-t/t_{gr}})}{s_{CO_2} \sum_{r=1}^{R} f_{CO_2 r}\, t_{CO_2 r}\, (1 - e^{-t/t_{CO_2 r}})} \tag{76}$$

"policy-maker" model gwp

$$GWP_g(t) = \frac{s_g\, b_g}{s_{CO_2}\, b_{CO_2}} \frac{\sum_{r=1}^{R} f_{gr}\, t_{gr}\, (1 - e^{-t/t_{gr}})}{\sum_{r=1}^{R} f_{CO_2 r}\, t_{CO_2 r}\, (1 - e^{-t/t_{CO_2 r}})}$$

Response to emissions in several periods

Additional concentration responses to emissions in several periods

$$\Delta r_{gr}(t) = \sum_{i=0}^{n} \Delta r_{gr}(t_{i-1}, t_i)\, \Phi_{gr}(t - t_i) + \Delta r_{gr}(t_n, t) \tag{99}$$

$$\Delta r_{gr}(t) = \left[\sum_{i=0}^{n} \Delta r_{gr}(t_{i-1}, t_i)\, \Phi_{gr}(t_n - t_i) \right] \Phi_{gr}(t - t_n) + \Delta r_{gr}(t_n, t) \tag{100}$$

$$\Delta r_g(t) = \Delta r_g(t_n) \sum_{r=1}^{R} f'_{gr}\, \Phi_{gr}(t - t_n) + \Delta r_g(t_n, t) \tag{101}$$

$$f'_{gr} = f_{gr} \frac{\sum_{i=0}^{n} \Delta r_{gr}(t_{i-1}, t_i)\, \Phi_{gr}(t_n - t_i)}{\Delta r_g(t_n)} \tag{102}$$

Temperature increase response to emissions in several periods

$$\Delta T_{grs}(t) = (t_{cs} / \bar{t}_c) \sum_{i=0}^{n} \Delta T_{grs}(t_{i-1}, t_i)\, \Theta_s(t - t_i) + \\ + \frac{K_g}{b_g} \sum_{i=0}^{n} \Delta r_{gr}(t_{i-1}, t_i)\, \Psi_{grs}(t - t_i) + \\ + \Delta T_{grs}(t_n, t) \tag{106}$$

$$\Delta T_{grs}(t) = (t_{cs} / \bar{t}_c) \left[\sum_{i=0}^{n} \left((t_{cs} / \bar{t}_c)\, \Delta T_{grs}(t_{i-1}, t_i)\, \Theta_s(t_n - t_i) + \frac{K_g}{b_g} \Delta r_{gr}(t_{i-1}, t_i)\, \Psi_{grs}(t_n - t_i) \right) \right] \Theta_s(t - t_n) \\ + \frac{K_g}{b_g} \left[\sum_{i=0}^{n} \Delta r_{gr}(t_{i-1}, t_i)\, \Phi_{gr}(t_n - t_i) \right] \Psi_{grs}(t - t_n) + \\ + \Delta T_{grs}(t_n, t) \tag{107}$$

$$\Delta T_g(t) = \Delta T_g(t_n) \sum_{s=1}^{S} l_s^{'}\ \Theta_s(t-t_n) +$$
$$+ \frac{K_g}{b_g} \Delta r_g(t_n) \sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr}^{'}\ \Psi_{grs}(t-t_n) + \quad (108)$$
$$+ \Delta T_g(t_n\ , t)$$

$$l_s^{'} = l_s \frac{(t_{cs}/\bar{t}_c)\sum_{r=1}^{R} f_{gr} \sum_{i=0}^{n}\left[(t_{cs}/\bar{t}_c)\Delta T_{grs}(t_{i-1}\ , t_i)\ \Theta_s(t_n - t_i) + \frac{K_g}{b_g} \Delta r_{gr}(t_{i-1}, t_i)\ \Psi_{grs}(t_n - t_i)\right]}{\Delta T_g(t_n)} \quad (109)$$

Additional concentration response to emissions over one period and afterwards

$$\Delta r_{gr}(t) = \Delta r_{gr}(t_a\ , t_b)\ \Phi_{gr}(t-t_b) + \Delta r_{gr}(t_b\ , t) \quad (110)$$

$$\Delta r_g(t) = \Delta r_g(t_b) \sum_{r=1}^{R} f_{gr}^{'}\ \Phi_{gr}(t-t_b) + \Delta r_g(t_b\ , t) \quad (111)$$

$$f_{gr}^{'} = f_{gr} \frac{\Delta r_{gr}(t_a\ , t_b)}{\Delta r_g(t_b)} \quad (112)$$

Temperature increase response to emissions over one period and afterwards

$$\Delta T_{grs}(t) = (t_{cs}/\bar{t}_c)\ \Delta T_{grs}(t_a\ , t_b)\ \Theta_s(t-t_b) + \frac{K_g}{b_g} \Delta r_{grs}(t_a\ , t_b)\ \Psi_{grs}(t-t_b) + \Delta T_g(t_b\ , t) \quad (113)$$

$$\Delta T_g(t) = \Delta T_g(t_b) \sum_{s=1}^{S} l_s^{'}\ \Theta_s(t-t_b) +$$
$$+ \frac{K_g}{b_g} \Delta r_g(t_b) \sum_{s=1}^{S} l_s \sum_{r=1}^{R} f_{gr}^{'}\ \Psi_{grs}(t-t_b) + \quad (114)$$
$$+ \Delta T_g(t_b\ , t)$$

$$l_s^{'} = l_s \frac{(t_{cs}/\bar{t}_c)\sum_{r=1}^{R} f_{gr}\ \Delta T_{grs}(t_a\ , t_b)}{\Delta T_g(t_b)} \quad (115)$$

8. Dimensionality of the variables

The dimensionality of the constants and functions in the note are as follows:

$[\Delta r\,] = [g]$

$[b\,] = 1$

$[e\,] = [\bar{e}\,] = [g\,][s]^{-1}$

$[e_0\,] = [g]$

$[cs] = [\mathrm{K}]$

$[\Delta T] = [\mathrm{K}]$

$[d\ \Delta T / d\ t\,] = [\mathrm{K}\,][s]^{-1}$

$[msl\,] = [cm]$

$[MSL] = [cm][\mathrm{K}]^{-1}$

$[\Delta r\,] = [\Delta \bar{r}\,] = [g]$

$[\Delta r_0\,] = [g\,][s]$

$[t\,] = [t\,] = [s]$

$[K] = [\mathrm{K}\,][s][g]^{-1}$

$[\Phi] = [\bar{\Phi}] = [\Theta] = [\bar{\Theta}] = [\Lambda] = [\bar{\Psi}] = [\bar{\Omega}] = 1$

$[\Psi] = [\Omega] = [\bar{\Lambda}] = [s]^{-1}$

$[\Gamma] = [g\,] = 1$

9. Example of application to data

The model adopted for the temperature response to a doubling of carbon dioxide concentration is (Voss,R. et al., in prep, Heinmann, M., personal communication):

$$\Delta T = 3.06K \left[1 - .634\, e^{-t/20y} - .366\, e^{-t/990y} \right]$$

for an initial concentration of carbon dioxide:

$$r_{CO_2 i} = 354.17\ ppmv$$

The pulse response of the additional concentration of carbon dioxide is taken from the "Bern" model (Joos *et al.*, 1996). Representative values are, for pulses of emission occurring at the time t_0:

t0 = 1770

$$\Delta r_{CO_2}(t) = .413 - .603 e^{-t/232.3y} + .501 e^{-t/100y} + .329 e^{-t/68.74y} - 4.705\, e^{-t/20y} + 4.988\, e^{-t/18.6} - .702 e^{-t/5.26y} + 2.377\, e^{-t/2.86y} - 2.083 e^{-t/2.18y} + .534\, e^{-t/1.27y}$$

t_0 = 1900

$$\Delta r_{CO_2}(t) = .237 + .653\, e^{-t/232.3y} - 1.963 e^{-t/100y} + 1.605\, e^{-t/68.74y} + .807\, e^{-t/20y} - .713\, e^{-t/18.6} + .444\, e^{-t/5.26y} - .739\, e^{-t/2.86y} + .792\, e^{-t/2.18y} - .127\, e^{-t/1.27y}$$

The single exponential decay time constants for all other greenhouse gases are taken from the IPCC Second Assessment Report.

The values of ($\bar{s}_g / \bar{s}_{CO_2}$) are taken from the 1995 IPCC Second Assessment report, in units of W/m^2 per ppmv, with the assumption that:

$$\bar{s}_g / \bar{s}_{CO_2} = s_g / s_{CO_2}$$

that is, the values, relative to carbon dioxide, of the constants sigma are the same for column and mean values.

The equivalence between the units of mass and volume fraction is taken to be .4636 ppmv/GtC for carbon dioxide; for other gases, this value is adjusted by the appropriate molecular mass.

The physical units of the variables are as follows:

- time in years (y);

- emissions in gigaton or petagram of carbon per year (GtC/y or PgC/y) for carbon dioxide; in teragram of nitrogen (TgN/y) for nitrous oxide; and in teragram of the gas (Tg_g/y) for all other greenhouse gases;

- pulse of emission in GtC for carbon dioxide; in TgN for nitrous oxide; and in Tg_g for all other greenhouse gases

- atmospheric concentration in parts per million in volume for carbon dioxide; and in parts per billion in volume for all other greenhouse gases;

- pulse of atmospheric concentration in ppmv.y for carbon dioxide; and in ppbv.y for all other greenhouse gases;

- temperature in degree Celsius (^oC);

- temperature rate of change in degree Celsius per year (^oC/y);

- mean sea level-rise in centimeter (cm).

The values of the constants in the formulas that define the response function, as well as the unit conversion constants appear in Table I, for 24 greenhouse gases included in the IPCC Second Assessment Report.

References

Joos, F.; Bruno, M.; Fink R.; Siegenthaler U.; Stocker T.F.; Le Quéré, C.; and Sarmiento, J.L. An efficient and accurate representation of complex oceanic and biospheric models of anthropogenic carbon uptake. Tellus **48B**, 397-147

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